Perturbations of Keplerian Orbits in Stationary Spherically Symmetric Spacetimes

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We study spherically symmetric perturbations determined by alternative theories of gravity to the gravitational field of a central mass in General Relativity. In particular, we focus on perturbations in the form of power laws, and calculate their effect on the secular variations of the orbital elements of a Keplerian orbit. We show that, to lowest approximation order, only the argument of pericentre and mean anomaly undergo secular variations; furthermore, we calculate the variation of the orbital period. We give analytic expressions for these variations which can be used to constrain the impact of alternative theories of gravity.

I. INTRODUCTION

In recent years, there has been much interest in studying theories of gravity alternative to General Relativity (GR): these theories have been motivated by the current observations, which seem to question the GR model of gravitational interactions on large scales, such as the galactic and cosmological ones. Just to mention some examples, one can consider $f(R)$ theories of gravity [1–3] or MOdified Newtonian Dynamics (MOND) [4]. Other theories alternative to GR are motivated by the attempts of giving a description of the gravitational interactions in a quantum framework: the recently proposed Hořava-Lifshitz gravity [5–7] is one example.

In order to get a deeper insight into these theories, it is important to test their predictions in a suitable weak-field and slow-motion limit: to this end, for instance, central mass solutions which generalize the one obtained in GR (the Schwarzschild solution) are investigated and their predictions compared to the data available from astronomical and astrophysical observations, in the Solar System and beyond. For these purposes, a general theoretical framework that can deal with classes of metric theories of gravity has been developed: the parametrized post-Newtonian (PPN) formalism. In the latter, the weak-field and slow-motion limit of these theories is studied in terms of suitable parameters, in such a way that experiments may fix their values: as a matter of fact, the current best estimates of the PPN parameters are in agreement with GR [8]. As a consequence, it is reasonable to believe that the effects of theories alternative to GR are small, so that they can be dealt with as perturbations of the GR background.

In this paper we want to suggest a simple approach that, without requiring the complex and comprehensive PPN framework, allows to evaluate the predictions of alternative theories of gravity on the Keplerian orbit of a test particle, which can be thought of as a simplified model of the dynamics of celestial bodies in planetary systems. In particular, we consider a general stationary and spherically symmetric (SSS) spacetime metric, that can be thought of a solution of the field equations of a generic alternative theory of gravity describing the gravitational field around a point-like central mass, and we work out the perturbations of the Keplerian orbital elements. We focus on perturbations in form of power laws, which have been recently considered in the literature [9]: these perturbations are interesting because they reproduce some known solutions of the field equations of alternative theories of gravity; moreover arbitrary spherically symmetric perturbations can be written in terms of power series, so that our results can be used quite generally. In particular, by using the Gauss perturbation scheme, we obtain, to lowest approximation order, the expressions of the secular variations of the orbital elements and the orbital period in terms of hypergeometric functions. These expressions can be used to place bounds on the parameters of alternative theories of gravity.

The paper is organized as follows: in Section II we write the perturbing acceleration in a generic SSS spacetime and obtain its expression to lowest approximation order, while the Gauss perturbation equations are briefly reviewed in Section III. In Section IV we focus on perturbations in the form of power laws, while conclusions are in Section V.

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II. THE PERTURBING ACCELERATION

We suppose that the gravitational field of a point mass in a generic alternative theory of gravity is described by the SSS spacetime metric[1]

$$ds^2 = (1+\phi(r))\,dt^2 - (1+\psi(r))\left(dr^2 + r^2 d\Omega^2\right) \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, and $\phi(r),\ \psi(r)$ are functions depending on the mass M of the source and, possibly, on other parameters of the theory. The metric (1) is written in isotropic polar coordinates, such that the spatial part of the metric is proportional to the flat spacetime metric $dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2$. Consequently, it is possible to write the metric (1) in Cartesian coordinates in the form

$$ds^2 = (1+\phi(r))\,dt^2 - (1+\psi(r))\left(dx^2 + dy^2 + dz^2\right) \tag{2}$$

where $r = \sqrt{x^2+y^2+z^2}$.

Since the effects of the generic alternative theory of gravity are expected to be small, we suppose that they can be considered as perturbations of the known GR solution, i.e. the Schwarzschild spacetime. This amounts to saying that $\phi(r)$ and $\psi(r)$ in some limit have to approach their GR values. In other words we suppose that in suitable limit, the metric takes the form

$$\phi(r) \ = \ \phi_{GR}(r) + \phi_A(r) \tag{3}$$
$$\psi(r) \ = \ \psi_{GR}(r) + \psi_A(r) \tag{4}$$

where the GR values are given by $\phi_{GR}(r) = -2M/r + 2M^2/r^2$, $\psi_{GR}(r) = 2M/r$ (see e.g. [10]) and the perturbations $\phi_A(r), \psi_A(r)$ due to the alternative gravity model are such that $\phi_A(r) \ll \phi_{GR}(r)$, $\psi_A(r) \ll \psi_{GR}(r)$.

In order to calculate the perturbations of the orbital elements, we must calculate the perturbing acceleration. To this end, first we assume that in the given theory, the matter is minimally and universally coupled, so that test particles follow geodesics of the metric (2) (or, equivalently (1)). Then, we consider the (post-Newtonian) equation of motion of a test particle (see [11])

$$\ddot{x}^i = -\frac{1}{2}h_{00,i} - \frac{1}{2}h_{ik}h_{00,k} + h_{00,k}\dot{x}^k\dot{x}^i + \left(h_{ik,m} - \frac{1}{2}h_{km,i}\right)\dot{x}^k\dot{x}^m \tag{5}$$

where "dot" stands for derivative with respect to the coordinate time. Since in our notation it is $h_{00} = \phi(r)$ and $h_{ij} = \psi(r)\delta_{ij}$, we can write the perturbing acceleration $\vec{\mathbf{W}}$ in the form

$$\vec{\mathbf{W}} = -\frac{1}{2}\left\{\Phi(r)\left[1+\psi_A(r)\right] + \Psi(r)v^2\right\}\hat{\mathbf{x}} + \left[\Phi(r)+\Psi(r)\right](\hat{\mathbf{x}}\cdot\vec{\mathbf{v}})\,\vec{\mathbf{v}} \tag{6}$$

where we set $\vec{\mathbf{x}} = (x,y,z)$, $\vec{\mathbf{v}} = (\dot{x},\dot{y},\dot{z})$, $\hat{\mathbf{x}} = \vec{\mathbf{x}}/|\vec{\mathbf{x}}|$, and

$$\Phi(r) \doteq \frac{d\phi_A(r)}{dr}, \qquad \Psi(r) \doteq \frac{d\psi_A(r)}{dr} \tag{7}$$

We aim at investigating the lowest order effects on planetary motion of $\phi_A(r)$ and $\psi_A(r)$: to this end, it is sufficient to apply the Gauss perturbation scheme to a Keplerian ellipse.

Moreover, since we are interested in the lowest order effects, we may also neglect the non linear terms (i.e. non linear perturbations with respect to flat spacetime). To this end, we start by noticing that to Newtonian order, it is $v^2 \simeq \phi_{GR}(r) \simeq M/r$. As a consequence, in (6) we may neglect the terms proportional to v^2 and to $(\hat{\mathbf{x}}\cdot\vec{\mathbf{v}})\,\vec{\mathbf{v}}$ (which is also proportional to the orbital eccentricity) and also the term $\Phi(r)\psi_A(r)$. In summary, in the weak-field and slow-motion limit the perturbing acceleration that we are going to consider is purely radial and is given by

$$\vec{\mathbf{W}} = -\frac{1}{2}\Phi(r)\hat{\mathbf{x}} \doteq W_r\hat{\mathbf{x}} \tag{8}$$

We point out that even though we started from the study of a SSS spacetime with the aim of setting bounds on alternative theories of gravity, what follows can be applied to generic radial perturbations (in particular, to those in form of a power law) of the orbital elements of a Keplerian motion.

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[1] If not otherwise stated, we use units such that $c = G = 1$; arrowed bold face letters like $\vec{\mathbf{x}}$ refer to spatial vectors while Latin indices refer to spatial components.

III. PERTURBATION EQUATIONS

We start from the expression (8) of the acceleration and apply the Gauss perturbation scheme to derive the secular variations. Because of spherical symmetry, the motion of test particles is confined to a plane and, in this plane, the unperturbed Keplerian ellipse is

$$r = \frac{a\left(1-e^2\right)}{1+e\cos f} \tag{9}$$

where a is the semimajor axis, e the eccentricity, f the true anomaly describing the particle angular distance from the pericentre. For purely radial perturbations, the Gauss equations for the variations of the Keplerian orbital elements read [12]

$$\frac{da}{dt} = \frac{2e}{n\sqrt{1-e^2}}W_r \sin f, \tag{10}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}W_r \sin f, \tag{11}$$

$$\frac{dI}{dt} = 0, \tag{12}$$

$$\frac{d\Omega}{dt} = 0, \tag{13}$$

$$\frac{d\omega}{dt} = -\frac{\sqrt{1-e^2}}{nae}W_r \cos f, \tag{14}$$

$$\frac{d\mathcal{M}}{dt} = n - \frac{2}{na}W_r\left(\frac{r}{a}\right) - \sqrt{1-e^2}\frac{d\omega}{dt}, \tag{15}$$

where $\mathcal{M}$ is mean anomaly, I is the orbital inclination, Ω is the ascending node, ω is the argument of pericentre, $n = \sqrt{M/a^3}$ is the Keplerian mean motion. In a Keplerian orbit the mean anomaly is a linear function of time defined by $\mathcal{M} = n\left(t - t_0\right)$, where t_0 is the time of a passage through pericentre. The orbital period P_b is related to the mean motion by $n = 2\pi/P_b$.

In order to obtain the secular effects we must evaluate the Gauss equations onto the unperturbed Keplerian ellipse (9), and then we must average them over one orbital period of the test particle. From (10) and (11), and taking into account (9), we see that the secular variations of the semimajor axis and eccentricity are null, because when averaging them the arguments of the integrals are odd functions. Hence, we obtain that in SSS spacetimes, to lowest approximation order (i.e. when the perturbations are purely radial) semimajor axis, eccentricity, beside the inclination and the node (which are only affected by normal i.e. out-of the-plane perturbations), are not affected by secular variations. We point out that, once that the secular variations of the argument of the pericentre $<\omega>$ and mean anomaly $<\mathcal{M}>$ have been determined, it is possible to obtain the corresponding variation of the mean longitude $<\lambda>\doteq<\omega>+<\mathcal{M}>$ [12].

Starting from the equation describing the variation of the mean anomaly, it is possible to evaluate the perturbation of the orbital period. In fact (see [13]), on using

$$df = \left(\frac{a}{r}\right)^2\sqrt{1-e^2}d\mathcal{M} \tag{16}$$

from (15) it is possible to write

$$\frac{df}{dt} = n\left(\frac{a}{r}\right)^2\sqrt{1-e^2}\left[1 - \frac{2}{n^2a}W_r\left(\frac{r}{a}\right) + \frac{1-e^2}{n^2ae}W_r\cos f\right] \tag{17}$$

Since we suppose that the perturbations are small, we may write

$$\frac{dt}{df} \simeq \left(\frac{r}{a}\right)^2\frac{1}{n\sqrt{1-e^2}}\left[1 + \frac{2}{n^2a}W_r\left(\frac{r}{a}\right) - \frac{1-e^2}{n^2ae}W_r\cos f\right] \tag{18}$$

And then, by integrating over one revolution along the unperturbed orbit, we obtain

$$P \simeq P_b + P_A \tag{19}$$

where

$$P_b \doteq \int_0^{2\pi} \left(\frac{r}{a}\right)^2 \frac{df}{n\sqrt{1-e^2}} = \frac{2\pi}{n} \tag{20}$$

is the unperturbed period, and

$$P_A \doteq \int_0^{2\pi} \left(\frac{r}{a}\right)^2 \left[\frac{2}{n^2 a} W_r \left(\frac{r}{a}\right) - \frac{1-e^2}{n^2 a e} W_r \cos f\right] \frac{df}{n\sqrt{1-e^2}} \tag{21}$$

is the variation of the orbital period due to the perturbation.

The above equations (14), (15) and (21) allow to calculate the variations of the argument of pericentre, mean anomaly and orbital period. This can be accomplished at least by numerical methods for arbitrary perturbations, however, as we are going to show in next Sextion, when the perturbation is in the form of a power law, it is possible to obtain analytic expressions.

IV. POWER LAWS PERTURBATIONS

After having discussed in the previous Section the expressions for the secular perturbations of the orbital elements for a generic purely radial perturbation, here we focus on the particular case of perturbations in the form of a power law. As we are going to show, in this case we can obtain analytic expressions for the secular variations. We point out that an arbitrary spherical symmetric perturbation that is expressed by an analytic function can be expanded in power series up to the required approximation level: hence, by knowing the contribution of each term of the series, it is possible to evaluate the whole effect of the perturbation, within the required accuracy. Eventually, even though our approach is motivated by the study of the effects of alternative theories of gravity, the results that we are going to obtain are quite general, and apply to radial perturbation of Keplerian orbits by means of arbitrary power laws.

In particular, we consider two kinds of perturbation that we write as follows. Given a constant α, which is a parameter deriving from the gravity model alternative to GR, the perturbations that we focus on are in the form: (i) $\phi_A(r) = \alpha r^N$ where the integer N is such that $N \leq -1$, or differently speaking $\phi_A(r) = \frac{\alpha}{r^{|N|}}, \quad |N| \geq 1$; (ii) $\phi_A(r) = \alpha r^N$ where the integer N is such that $N \geq 1$. The perturbations of the first kind are asymptotically flat, while the second ones are not: in the latter case, we assume that there exists a range $0 < r < \bar{r}$ for which $\phi_A(r) \ll \phi_{GR}(r)$, and our results are valid within this range. We notice that, for a logarithmic perturbation in the form $\phi_A(r) = \beta \log(r/r_0)$, it is $W_r = -\frac{1}{2}\frac{\beta}{r}$, which can be dealt with as in the case (i) above.

The expressions of the secular variations that we give below can be used to place constraints on α by means of a comparison with the available data.

A. Perturbation of the argument of pericentre

For perturbations in the form $\phi_A(r) = \frac{\alpha}{r^{|N|}}, \quad |N| \geq 1$, we start from eq. (14), and we average it over one orbital period by taking into account the expression of the unperturbed orbit (9) and the relation [14]

$$dt = \frac{(1-e^2)^{3/2}}{n(1+e\cos f)^2} df \tag{22}$$

On using (A3), we then obtain

$$< \omega >= -\frac{1}{2} \frac{\pi \alpha |N| (|N|-1) \left(1-e^2\right)^{1-|N|}}{n^2 a^{2+|N|}} F\left(1 - \frac{|N|}{2}, \frac{3}{2} - \frac{|N|}{2}, 2, e^2\right) \tag{23}$$

where F is the hypergeometric function (see Appendix A). In particular, this result is in agreement with the correspondent expression found in [9], where the effects of a central force were considered.

On the other hand, for perturbations in the form $\phi_A(r) = \alpha r^N$ with $N \geq 1$, we start again from (14) but, in order to average it over one orbital, it is useful to introduce the expression of the unperturbed orbit

$$r = a(1 - e\cos E) \tag{24}$$

in terms of the eccentric anomaly E, which is is related to the true anomaly f by the following relations:

$$\cos f = \frac{\cos E - e}{1 - e\cos E}, \qquad \sin f = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}. \tag{25}$$

On using the relation [14]

$$dt = \frac{1}{n}\left(1 - e\cos E\right) dE \tag{26}$$

and the integrals (A3) and (A4), we obtain

$$< \omega >= -\frac{1}{2}\frac{\pi\alpha N\sqrt{1-e^2}}{n^2 a^{2-N}}\left[\left(N-1\right)F\left(1-\frac{N}{2},\frac{3}{2}-\frac{N}{2},2,e^2\right)+2F\left(1-\frac{N}{2},\frac{1}{2}-\frac{N}{2},1,e^2\right)\right] \tag{27}$$

or, equivalently

$$< \omega >= -\frac{1}{2}\frac{\pi\alpha N\left(N+1\right)\sqrt{1-e^2}}{n^2 a^{2-N}}F\left(1-\frac{N}{2},\frac{1}{2}-\frac{N}{2},2,e^2\right) \tag{28}$$

In particular, eq. (28) is in agreement with the correspondent expression found in [9].

These general expressions can be used to reproduce known results. For instance, on setting $N = 1$ in eqs (28) we obtain the variation of pericentre due to a constant perturbation acceleration

$$< \omega >= -\frac{\pi\alpha}{n^2 a}\sqrt{1-e^2} \tag{29}$$

This result is in agreement with Sanders [15] (also reported by [9]), who, working in MOND framework, considered the case of the constant acceleration to put constraints on the effects of anomalous acceleration by analyzing Mercury's advance of perihelion.

The perturbation in the form $\phi_A(r) = \alpha r^2$ coincides with the effect of a cosmological constant Λ [16],[17] in GR, while in the case of $f(R)$ gravity, it describes the vacuum solution in Palatini formalism [18, 19]. In this case, by setting $N = 2$ in eq. (28) we get

$$< \omega >= -\frac{3\pi\alpha}{n^2}\sqrt{1-e^2} \tag{30}$$

On setting $\alpha = -\frac{1}{3}\Lambda$ and using the relation $n^2 a^3 = M$ which holds for the unperturbed orbit, eq. (30) becomes

$$< \omega >= \frac{\pi\Lambda a^3}{M}\sqrt{1-e^2} \tag{31}$$

in agreement with [16] and [19].

On setting $|N| = 4$ in (23), we obtain the perturbation determined by a vacuum solution of Hořava-Lifshitz gravity [20, 21]; by approximating the hypergeometric function, the variation of the argument of pericentre becomes

$$< \omega >\simeq -\frac{6\pi\alpha}{n^2 a^6}\frac{\left(1+\frac{1}{4}e^2\right)}{\left(1-e^2\right)^3} \tag{32}$$

in agreement with [20].

Eventually, we consider a logarithmic perturbation, in the form $\phi_A(r) = \beta\log(r/r_0)$. From eq. (23), by approximation up to e^4 we obtain

$$< \omega >= -\frac{1}{2}\frac{\pi\beta\left(1-e^2\right)}{n^2 a^2}F\left(1,\frac{3}{2},2,e^2\right) \simeq -\frac{1}{2}\frac{\pi\beta\left(1-e^2\right)}{n^2 a^2}\left(1-\frac{1}{4}e^2\right) \tag{33}$$

in agreement with [9].

B. Perturbation of the mean anomaly

We may proceed as in the previous section to calculate the secular variation of the mean anomaly. For a perturbation in the form $\phi_A(r) = \frac{\alpha}{r^{|N|}}, \quad |N| \geq 1$, we start from eq. (15), and focus on the part that is proportional to W_r; after averaging over the unperturbed ellipse (9) making use of eqs. (22) and (A4), we get

$$< \mathcal{M} >= -\frac{2\pi\alpha|N|\left(1-e^2\right)^{3/2-|N|}}{n^2a^{2+|N|}}F\left(\frac{1}{2}-\frac{|N|}{2},-\frac{|N|}{2},1,e^2\right)-\sqrt{1-e^2}< \omega > \tag{34}$$

where $< \omega >$ is given by (23).

Similarly, for perturbations in the form $\phi_A(r) = \alpha r^N$, with $N \geq 1$, averaging the part of (15) which depens on W_r, making use of (26) and (A4), we obtain

$$< \mathcal{M} >= \frac{2\pi\alpha N}{n^2a^{2-N}}F\left(-\frac{N}{2},-\frac{1}{2}-\frac{N}{2},1,e^2\right)-\sqrt{1-e^2}< \omega > \tag{35}$$

where $< \omega >$ is given by (28).

On setting $N = 2$, from eq. (35), by approximating the hypergeometric function, the variation of the mean anomaly is

$$< \mathcal{M} >= \frac{4\pi\alpha}{n^2}F\left(-1,-\frac{3}{2},1,e^2\right)+\frac{3\pi\alpha}{n^2}\left(1-e^2\right) \simeq \frac{3\pi\alpha}{n^2}\left(\frac{7}{3}+e^2\right) \tag{36}$$

in agreement with [19].

C. Perturbation of the orbital period

Starting from eq. (21), we may calculate the variation of the orbital period. In particular, for a perturbation in the form $\phi_A(r) = \frac{\alpha}{r^{|N|}}, \quad |N| \geq 1$, on evaluating eq. (21) over the unperturbed ellipse (9) making use of the integrals (A3), (A4), we obtain

$$P_A = \frac{\pi\alpha|N|\left(1-e^2\right)^{3/2-|N|}}{n^3a^{2+|N|}}\left[2F\left(1-\frac{|N|}{2},\frac{3}{2}-\frac{|N|}{2},1,e^2\right)-\frac{1}{2}\left(|N|-1\right)F\left(1-\frac{|N|}{2},\frac{3}{2}-\frac{|N|}{2},2,e^2\right)\right] \tag{37}$$

As for perturbations in the form $\phi_A(r) = \alpha r^N$ with $N \geq 1$, on evaluating eq. (21) over the unperturbed ellipse (24) making use of the relation (16) and the integrals (A3), (A4), we obtain

$$P_A = -\frac{\pi\alpha N}{n^3a^{2-N}}\left[2F\left(-\frac{N}{2},-\frac{1}{2}-\frac{N}{2},1,e^2\right)+\frac{\left(1-e^2\right)}{2}\left(N+1\right)F\left(1-\frac{N}{2},\frac{1}{2}-\frac{N}{2},2,e^2\right)\right] \tag{38}$$

On setting $|N| = 4$, from eq. (37), by approximating the hypergeometric functions, we obtain

$$P_A \simeq \frac{\pi\alpha\left(1-e^2\right)^{-5/2}}{n^3a^6}\left(2+\frac{5}{2}e^2\right) \tag{39}$$

in agreement with [21].

Furthermore, on setting $N = 2$ in (38), by approximation of the hypergeometric functions, the perturbation of the orbital period turns out to be

$$P_A \simeq -\frac{\pi\alpha}{n^3}\left(7+3e^2\right) \tag{40}$$

in agreement with [17].

V. CONCLUSIONS

We considered a general stationary and spherically symmetric spacetime, and worked out the perturbations determined by a generic alternative theory of gravity to the GR solution describing the gravitational field around a central mass. In order to evaluate the effects of such perturbations, we showed that, in the weak-field and slow-motion limit, to lowest approximation order, these effects can be described by a purely radial acceleration. Then, we considered the Keplerian orbit of a test particle, which can be thought of as a model of the dynamics of celestial bodies in planetary systems, and we evaluated the impact of such perturbations on the orbital elements.

In particular, we obtained analytic expressions, in terms of hypergeometric functions, for the secular variations of the advance of pericentre, mean anomaly and orbital period determined by perturbations in form of power laws; the other orbital elements do not undergo secular variations to lowest approximation order. The expressions for the variation of the argument of pericentre are in agreement with the results recently obtained, pertaining to the orbital precession due to central force perturbations.

Our results are quite general, since even though were motivated by the study of the effect of alternative theories of gravity, they can be applied to arbitrary radial power laws perturbations of the orbital elements of a Keplerian motion.

Spherically symmetric perturbations in the form of a power law were obtained both in GR, for instance when the effect of a cosmological constant is considered, and in alternative theories such as $f(R)$, MOND, Hořava-Lifshitz gravity; we have shown that our results are in agreement with those already available in the literature pertaining to such alternative theories. For an arbitrary perturbation it is possible to obtain the secular variations by means of numerical approaches, or by expanding it in power series and applying our results to each term of the series.

The possibility of testing the predictions of alternative theories of gravity on planetary motion is important to verify their reliability on scales different from those typical of galactic dynamics or cosmology, where they are usually tested. The simple approach that we considered here allows to place bounds on the theory parameters, by evaluating their impact on the Keplerian dynamics and allowing a comparison with the available data.

Appendix A: Solutions of Integrals by means of Hypergeometric Functions

In order to evaluate the secular variations of the Keplerian elements and of the orbital period determined by the perturbations in the form of power laws that we have considered, it is necessary to solve the following integrals

$$I_N = \int_0^{2\pi} \cos u \left[1 + e\cos u\right]^N du \tag{A1}$$

$$L_N = \int_0^{2\pi} \left[1 - e\cos u\right]^N du \tag{A2}$$

They can be solved by subsituting $\cos u = z$, and then using the binomial theorem for $(1+z)^N$. The solutions are given in terms of hypergeometric functions:

$$I_N = \pi N e F\left(\frac{1}{2} - \frac{N}{2}, 1 - \frac{N}{2}, 2, e^2\right) \tag{A3}$$

$$L_N = 2\pi F\left(-\frac{N}{2}, \frac{1}{2} - \frac{N}{2}, 1, e^2\right) \tag{A4}$$

where $F(a,b,c,x) \doteq_2 F_1(a,b,c,d)$ defined by (see e.g. [22])

$${}_2F_1(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{A5}$$

where the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n \doteq \frac{(a+n-1)!}{(a-1)!} \tag{A6}$$

$$(a)_0 \doteq 1 \tag{A7}$$

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